

## Some conjectures on congruences

ZHI-HONG SUN

School of Mathematical Sciences, Huaiyin Normal University,  
Huaian, Jiangsu 223001, PR China

Email: zhihongsun@yahoo.com

Homepage: <http://www.hytc.edu.cn/xsjl/szh>

ABSTRACT. Let  $p$  be an odd prime. In the paper we collect the author's various conjectures on congruences modulo  $p$  or  $p^2$ , which are concerned with sums of binomial coefficients, Lucas sequences, power residues and special binary quadratic forms.

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### 1. Notation.

Let  $\mathbb{Z}$  and  $\mathbb{N}$  be the sets of integers and positive integers respectively. For  $b, c \in \mathbb{Z}$  the Lucas sequences  $\{U_n(b, c)\}$  and  $\{V_n(b, c)\}$  are defined by

$$(1.1) \quad \begin{aligned} U_0(b, c) &= 0, \quad U_1(b, c) = 1, \\ U_{n+1}(b, c) &= bU_n(b, c) - cU_{n-1}(b, c) \quad (n \geq 1) \end{aligned}$$

and

$$(1.2) \quad \begin{aligned} V_0(b, c) &= 2, \quad V_1(b, c) = b, \\ V_{n+1}(b, c) &= bV_n(b, c) - cV_{n-1}(b, c) \quad (n \geq 1). \end{aligned}$$

Let  $d = b^2 - 4c$ . It is well known that for  $n \in \mathbb{N}$ ,

$$(1.3) \quad U_n(b, c) = \begin{cases} \frac{1}{\sqrt{d}} \left\{ \left( \frac{b+\sqrt{d}}{2} \right)^n - \left( \frac{b-\sqrt{d}}{2} \right)^n \right\} & \text{if } d \neq 0, \\ n \left( \frac{b}{2} \right)^{n-1} & \text{if } d = 0 \end{cases}$$

and

$$(1.4) \quad V_n(b, c) = \left( \frac{b+\sqrt{d}}{2} \right)^n + \left( \frac{b-\sqrt{d}}{2} \right)^n.$$

Let  $[x]$  be the greatest integer not exceeding  $x$ , and let  $\left(\frac{a}{m}\right)$  be the Jacobi symbol. For  $m \in \mathbb{Z}$  with  $m = 2^\alpha m_0 (2 \nmid m_0)$  we say that  $2^\alpha \parallel m$  and  $m_0$  is the odd part of  $m$ . For  $t \in \mathbb{Z}$  let

$$\delta(t) = \begin{cases} 1 & \text{if } 8 \mid t, \\ -1 & \text{if } 8 \nmid t. \end{cases}$$

For an integer  $m$  and odd prime  $p$  with  $p \nmid m$  let

$$Z_p(m) = \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{m^n} \sum_{k=0}^n \binom{n}{k}^3.$$

## 2. Conjectures on power residues.

In 1980 and 1984 Hudson and Williams proved the following result.

**Theorem 2.1.** *Let  $p \equiv 1 \pmod{24}$  be a prime and hence  $p = c^2 + d^2 = x^2 + 3y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$ .*

- (i) ([HW]) *If  $c \equiv \pm(-1)^{\frac{y}{4}} \pmod{3}$ , then  $3^{\frac{p-1}{8}} \equiv \pm 1 \pmod{p}$ .*
- (ii) ([H]) *If  $d \equiv \pm(-1)^{\frac{y}{4}} \pmod{3}$ , then  $3^{\frac{p-1}{8}} \equiv \pm \frac{d}{c} \pmod{p}$ .*

Hudson and Williams proved Theorem 2.1(i) by using the cyclotomic numbers of order 12, and Hudson proved Theorem 2.1(ii) using the Jacobi sums of order 24.

In [S3] the author posed the following conjectures similar to Theorem 2.1.

**Conjecture 2.1 ([S3, Conjecture 9.1]).** *Let  $p \equiv 13 \pmod{24}$  be a prime and hence  $p = c^2 + d^2 = x^2 + 3y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$ ,  $x = 2^\alpha x_0$ ,  $y = 2^\beta y_0$  and  $x_0 \equiv y_0 \equiv 1 \pmod{4}$ . Then*

$$3^{\frac{p-5}{8}} \equiv \begin{cases} \pm \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm c \pmod{3}, \\ \mp \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm d \pmod{3}. \end{cases}$$

Conjecture 2.1 has been checked for all primes  $p < 20,000$ .

**Conjecture 2.2 ([S3, Conjecture 9.2]).** *Let  $p \equiv 1, 9, 25 \pmod{28}$  be a prime and hence  $p = c^2 + d^2 = x^2 + 7y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$ ,  $x = 2^\alpha x_0$ ,  $y = 2^\beta y_0$  and  $x_0 \equiv y_0 \equiv 1 \pmod{4}$ .*

- (i) *If  $p \equiv 1 \pmod{8}$ , then*

$$7^{\frac{p-1}{8}} \equiv \begin{cases} -(-1)^{\frac{y}{4}} \pmod{p} & \text{if } 7 \mid c, \\ (-1)^{\frac{y}{4}} \pmod{p} & \text{if } 7 \mid d, \\ \mp (-1)^{\frac{y}{4}} \frac{d}{c} \pmod{p} & \text{if } c \equiv \pm d \pmod{7}. \end{cases}$$

- (ii) *If  $p \equiv 5 \pmod{8}$ , then*

$$7^{\frac{p-5}{8}} \equiv \begin{cases} -\frac{y}{x} \pmod{p} & \text{if } 7 \mid c, \\ \frac{y}{x} \pmod{p} & \text{if } 7 \mid d, \\ \mp \frac{dy}{cx} \pmod{p} & \text{if } c \equiv \pm d \pmod{7}. \end{cases}$$

Conjecture 2.2 has been checked for all primes  $p < 20,000$ .

**Conjecture 2.3** ([S3, Conjecture 9.7]). *Let  $p \equiv 1, 9 \pmod{20}$  be a prime and hence  $p = c^2 + d^2 = x^2 + 5y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are congruent to 1 modulo 4.*

(i) *If  $p \equiv 1 \pmod{8}$ , then*

$$5^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{d}{4}} \delta(y) \pmod{p} & \text{if } x \equiv \pm c \pmod{5}, \\ \pm(-1)^{\frac{d}{4}} \delta(y) \frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{5}. \end{cases}$$

(ii) *If  $p \equiv 5 \pmod{8}$ , then*

$$5^{\frac{p-5}{8}} \equiv \begin{cases} \pm\delta(x) \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm c \pmod{5}, \\ \mp\delta(x) \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm d \pmod{5}. \end{cases}$$

Conjecture 2.3 has been checked for all primes  $p < 20,000$ .

**Conjecture 2.4** ([S3, Conjecture 9.8]). *Let  $p \equiv 1, 9 \pmod{40}$  be a prime and hence  $p = c^2 + d^2 = x^2 + 10y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv x \equiv 1 \pmod{4}$ . Then*

$$5^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{d}{4} + \frac{x-1}{4}} \frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{5}, \\ \pm(-1)^{\frac{d}{4} + \frac{x-1}{4}} \pmod{p} & \text{if } x \equiv \pm c \pmod{5}. \end{cases}$$

**Conjecture 2.5** ([S3, Conjecture 9.9]). *Let  $p \equiv 1, 9, 17, 25, 29, 49 \pmod{52}$  be a prime and hence  $p = c^2 + d^2 = x^2 + 13y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are congruent to 1 modulo 4.*

(i) *If  $p \equiv 1 \pmod{8}$ , then*

$$13^{\frac{p-1}{8}} \equiv \begin{cases} \mp(-1)^{\frac{d}{4}} \delta(y) \frac{d}{c} \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 1, \pm 3, \pm 9 \pmod{13}, \\ \pm(-1)^{\frac{d}{4}} \delta(y) \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 2, \pm 5, \pm 6 \pmod{13}. \end{cases}$$

(ii) *If  $p \equiv 5 \pmod{8}$ , then*

$$13^{\frac{p-5}{8}} \equiv \begin{cases} \pm\delta(x) \frac{y}{x} \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 1, \pm 3, \pm 9 \pmod{13}, \\ \pm\delta(x) \frac{dy}{cx} \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 2, \pm 5, \pm 6 \pmod{13}. \end{cases}$$

**Conjecture 2.6** ([S3, Conjecture 9.16]). *Let  $p \equiv 1 \pmod{4}$  be a prime and  $p = c^2 + d^2 = x^2 + 17y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are numbers of the form  $4k + 1$ .*

(i) *If  $p \equiv 1 \pmod{8}$ , then*

$$17^{\frac{p-1}{8}} \equiv \begin{cases} -(-1)^{\frac{d}{4} + \frac{xy}{4}} \frac{d}{c} \pmod{p} & \text{if } 4c + d \equiv \pm 6x, \pm 7x \pmod{17}, \\ (-1)^{\frac{d}{4} + \frac{xy}{4}} \frac{d}{c} \pmod{p} & \text{if } 4c + d \equiv \pm 3x, \pm 5x \pmod{17}, \\ (-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } 4c + d \equiv \pm x, \pm 4x \pmod{17}, \\ -(-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } 4c + d \equiv \pm 2x, \pm 8x \pmod{17}. \end{cases}$$

(ii) If  $p \equiv 5 \pmod{8}$ , then

$$17^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^x \frac{y}{x} \pmod{p} & \text{if } 4c + d \equiv \pm 6x, \pm 7x \pmod{17}, \\ -(-1)^x \frac{y}{x} \pmod{p} & \text{if } 4c + d \equiv \pm 3x, \pm 5x \pmod{17}, \\ (-1)^x \frac{dy}{cx} \pmod{p} & \text{if } 4c + d \equiv \pm x, \pm 4x \pmod{17}, \\ -(-1)^x \frac{dy}{cx} \pmod{p} & \text{if } 4c + d \equiv \pm 2x, \pm 8x \pmod{17}. \end{cases}$$

**Conjecture 2.7 ([S5, Conjecture 4.3]).** Let  $p \equiv 1 \pmod{4}$  and  $q \equiv 3 \pmod{8}$  be primes such that  $p = c^2 + d^2 = x^2 + qy^2$  with  $c, d, x, y \in \mathbb{Z}$  and  $q \mid cd$ . Suppose  $c \equiv 1 \pmod{4}$ ,  $x = 2^\alpha x_0$ ,  $y = 2^\beta y_0$  and  $x_0 \equiv y_0 \equiv 1 \pmod{4}$ .

(i) If  $p \equiv 1 \pmod{8}$ , then

$$q^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{y}{4}} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp(-1)^{\frac{q-3}{8} + \frac{y}{4}} \frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}. \end{cases}$$

(ii) If  $p \equiv 5 \pmod{8}$ , then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} \pm \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp(-1)^{\frac{q-3}{8}} \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}. \end{cases}$$

**Conjecture 2.8 ([S5, Conjecture 4.4]).** Let  $p \equiv 1 \pmod{4}$  and  $q \equiv 7 \pmod{16}$  be primes such that  $p = c^2 + d^2 = x^2 + qy^2$  with  $c, d, x, y \in \mathbb{Z}$  and  $q \mid cd$ . Suppose  $c \equiv 1 \pmod{4}$ ,  $x = 2^\alpha x_0$ ,  $y = 2^\beta y_0$  and  $x_0 \equiv y_0 \equiv 1 \pmod{4}$ .

(i) If  $p \equiv 1 \pmod{8}$ , then

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{y}{4}} \pmod{p} & \text{if } q \mid d, \\ -(-1)^{\frac{y}{4}} \pmod{p} & \text{if } q \mid c. \end{cases}$$

(ii) If  $p \equiv 5 \pmod{8}$ , then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} \frac{y}{x} \pmod{p} & \text{if } q \mid d, \\ -\frac{y}{x} \pmod{p} & \text{if } q \mid c. \end{cases}$$

**Conjecture 2.9 ([S5, Conjecture 4.5]).** Let  $p \equiv 1 \pmod{4}$  and  $q \equiv 15 \pmod{16}$  be primes such that  $p = c^2 + d^2 = x^2 + qy^2$  with  $c, d, x, y \in \mathbb{Z}$  and  $q \mid cd$ . Suppose  $c \equiv 1 \pmod{4}$ ,  $x = 2^\alpha x_0$ ,  $y = 2^\beta y_0$  and  $x_0 \equiv y_0 \equiv 1 \pmod{4}$ .

(i) If  $p \equiv 1 \pmod{8}$ , then  $q^{\frac{p-1}{8}} \equiv (-1)^{\frac{y}{4}} \pmod{p}$ .

(ii) If  $p \equiv 5 \pmod{8}$ , then  $q^{\frac{p-5}{8}} \equiv \frac{y}{x} \pmod{p}$ .

Conjectures 2.7-2.9 have been checked for all primes  $p < 100,000$  and  $q < 100$ .

**Conjecture 2.10** ([S3, Conjecture 9.5]). *Let  $p \equiv 1 \pmod{4}$  be a prime,  $b \in \mathbb{Z}$ ,  $2 \nmid b$ ,  $p \neq b^2 + 4$  and  $p = c^2 + d^2 = x^2 + (b^2 + 4)y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are numbers of the form  $4k + 1$ .*

(i) *If  $4 \nmid xy$ , then*

$$\begin{aligned} \left(\frac{b + \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} &\equiv -\left(\frac{b - \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} \\ &\equiv \begin{cases} -(-1)^{\frac{d}{4}} \frac{d}{c} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 1, 3 \pmod{8}, \\ (-1)^{\frac{d}{4}} \frac{d}{c} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 5, 7 \pmod{8}, \\ 1 \pmod{p} & \text{if } 2 \parallel y. \end{cases} \end{aligned}$$

(ii) *If  $4 \mid xy$ , then*

$$\begin{aligned} \left(\frac{b + \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} &\equiv \left(\frac{b - \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} \\ &\equiv \begin{cases} (-1)^{\frac{d+y}{4}} \pmod{p} & \text{if } 4 \mid y, \\ -(-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 1, 3 \pmod{8}, \\ (-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

We remark that Conjectures 2.1-2.10 have been solved by the author under some restricted conditions in [S9] and [S10].

### 3. Conjectures on Lucas sequences.

**Conjecture 3.1** ([S3, Conjecture 9.4]). *Let  $p \equiv 1 \pmod{4}$  be a prime,  $b \in \mathbb{Z}$ ,  $2 \nmid b$  and  $p = c^2 + d^2 = x^2 + (b^2 + 4)y^2 \neq b^2 + 4$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are numbers of the form  $4k + 1$ .*

(i) *If  $4 \nmid xy$ , then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} (-1)^{\frac{d}{4}} \frac{2y}{x} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 1, 3 \pmod{8}, \\ -(-1)^{\frac{d}{4}} \frac{2y}{x} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 5, 7 \pmod{8}, \\ \frac{2dy}{cx} \pmod{p} & \text{if } 2 \parallel y. \end{cases}$$

(ii) *If  $4 \mid xy$ , then*

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 2(-1)^{\frac{d+y}{4}} \pmod{p} & \text{if } 4 \mid y, \\ -2(-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 1, 3 \pmod{8}, \\ 2(-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 5, 7 \pmod{8}. \end{cases}$$

Conjecture 3.1 has been checked for  $1 \leq b < 60$  and  $p < 20,000$ . When  $p \equiv 1 \pmod{8}$ ,  $b = 1, 3$  and  $4 \mid y$ , the conjecture  $V_{\frac{p-1}{4}}(b, -1) \equiv 2(-1)^{\frac{d+y}{4}} \pmod{p}$  is equivalent to a conjecture of E. Lehmer. See [L, Conjecture 4].

By (1.3) and (1.4), Conjecture 3.1 is equivalent to Conjecture 2.10. Conjectures 2.10 and 3.1 were partially settled by the author in [S10]. By [S3], Conjectures 2.3 and 2.5 are consequences of Conjecture 3.1.

**Conjecture 3.2** ([S3, Conjecture 9.11]). *Let  $p \equiv 1 \pmod{4}$  be a prime,  $b \in \mathbb{Z}$ ,  $b \equiv 4 \pmod{8}$ ,  $p \neq b^2/4 + 1$  and  $p = c^2 + d^2 = x^2 + (1 + b^2/4)y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are numbers of the form  $4k + 1$ . Then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} (-1)^{\frac{b+4}{8} + \frac{d}{4} \frac{y}{x}} \pmod{p} & \text{if } 2 \parallel x, \\ \frac{dy}{cx} \pmod{p} & \text{if } 2 \parallel y, \\ 0 \pmod{p} & \text{if } 4 \mid xy \end{cases}$$

and

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 2(-1)^{\frac{d}{4} + \frac{y}{4}} \pmod{p} & \text{if } 4 \mid y, \\ 2(-1)^{\frac{b-4}{8} + \frac{x}{4} \frac{d}{c}} \pmod{p} & \text{if } 4 \mid x, \\ 0 \pmod{p} & \text{if } 4 \nmid xy. \end{cases}$$

**Conjecture 3.3** ([S3, Conjecture 9.14]). *Let  $p \equiv 1 \pmod{4}$  be a prime,  $b \in \mathbb{Z}$ ,  $8 \mid b$ ,  $p \neq b^2/4 + 1$  and  $p = c^2 + d^2 = x^2 + (1 + b^2/4)y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$  and all the odd parts of  $d, x, y$  are numbers of the form  $4k + 1$ . Then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 0 \pmod{p} & \text{if } 4 \mid xy, \\ -(-1)^{(\frac{b}{8}-1)y} \frac{dy}{cx} \pmod{p} & \text{if } 4 \nmid xy \end{cases}$$

and

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 2(-1)^{\frac{d}{4} + \frac{xy}{4} + \frac{b}{8}y} \pmod{p} & \text{if } 4 \mid xy, \\ 0 \pmod{p} & \text{if } 4 \nmid xy. \end{cases}$$

Conjectures 3.2 and 3.3 have been checked for  $1 \leq b < 100$  and  $p < 20,000$ . In [S10] the author partially solved the two conjectures. We mention that Conjecture 3.3 implies Conjecture 2.6.

**Conjecture 3.4** ([S3, Conjecture 9.17]). *Let  $p \equiv 1 \pmod{4}$  be a prime,  $b \in \mathbb{Z}$ ,  $b \equiv 2 \pmod{4}$ ,  $p \neq b^2/4 + 1$  and  $p = c^2 + d^2 = x^2 + (1 + b^2/4)y^2$  for some  $c, d, x, y \in \mathbb{Z}$ . Suppose  $c \equiv 1 \pmod{4}$ ,  $x = 2^\alpha x_0$ ,  $y = 2^\beta y_0$  and  $x_0 \equiv y_0 \equiv 1 \pmod{4}$ . Then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} (-1)^{\frac{b-2}{4} + \frac{d}{4} \frac{y}{x}} \pmod{p} & \text{if } 2 \parallel y, \\ 0 \pmod{p} & \text{if } 4 \mid y \end{cases}$$

and

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 0 \pmod{p} & \text{if } 2 \parallel y, \\ 2(-1)^{\frac{d}{4} + \frac{y}{4}} \pmod{p} & \text{if } 4 \mid y. \end{cases}$$

Conjecture 3.4 has been checked for  $1 \leq b < 100$  and  $p < 20,000$ . By [S3], Conjecture 3.4 implies Conjecture 2.4.

**Conjecture 3.5** ([S5, Conjecture 4.1]). *Let  $p$  be an odd prime and  $k \in \mathbb{N}$  with  $2 \nmid k$ . Suppose  $p = x^2 + (k^2 + 1)y^2$  for some  $x, y \in \mathbb{Z}$ . Then*

$$V_{\frac{p+1}{4}}(2k, -1) \equiv \begin{cases} (-1)^{\frac{(\frac{p-1}{2}y)^2-1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 5, 7 \pmod{8}, \\ -(-1)^{\frac{(\frac{p-1}{2}y)^2-1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 1, 3 \pmod{8}. \end{cases}$$

Conjecture 3.5 has been checked for all  $k < 60$  and  $p < 20,000$ . The case  $k = 1$  has been solved by the author in [S3].

**Conjecture 3.6** ([S5, Conjecture 4.2]). *Let  $p$  be an odd prime and  $k \in \mathbb{N}$  with  $2 \nmid k$ . Suppose  $2p = x^2 + (k^2 + 4)y^2$  for some  $x, y \in \mathbb{Z}$ .*

(i) *If  $k \equiv 1, 3 \pmod{8}$ , then*

$$V_{\frac{p+1}{4}}(k, -1) \equiv \begin{cases} (-1)^{\frac{(\frac{p-1}{2}y)^2-1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 1, 11 \pmod{16}, \\ -(-1)^{\frac{(\frac{p-1}{2}y)^2-1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 3, 9 \pmod{16}. \end{cases}$$

(ii) *If  $k \equiv 5, 7 \pmod{8}$ , then*

$$V_{\frac{p+1}{4}}(k, -1) \equiv \begin{cases} (-1)^{\frac{(\frac{p-1}{2}y)^2-1}{8}} 2^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 5, 15 \pmod{16}, \\ -(-1)^{\frac{(\frac{p-1}{2}y)^2-1}{8}} 2^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 7, 13 \pmod{16}. \end{cases}$$

Conjecture 3.6 has been checked for all  $k < 60$  and  $p < 20,000$ . The case  $k = 1$  was posed by the author in [S1] and solved by C.N. Beli in [B].

**Conjecture 3.7** ([S6, Conjecture 5.1]). *Let  $p > 7$  be a prime such that  $p \equiv 1, 2, 4 \pmod{7}$  and so  $p = C^2 + 7D^2$  with  $C, D \in \mathbb{Z}$ .*

(i) *If  $p \equiv 1 \pmod{4}$  and  $C \equiv 1 \pmod{4}$ , then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 U_k(16, 1) &\equiv 0 \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 V_k(16, 1) &\equiv (-1)^{\frac{p-1}{4}} \left(4C - \frac{p}{C}\right) \pmod{p^2}. \end{aligned}$$

(ii) *If  $p \equiv 3 \pmod{4}$  and  $D \equiv 1 \pmod{4}$ , then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 U_k(16, 1) &\equiv (-1)^{\frac{p-3}{4}} \left(\frac{16}{3}D - \frac{4p}{21D}\right) \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 V_k(16, 1) &\equiv (-1)^{\frac{p+1}{4}} \left(84D - \frac{3p}{D}\right) \pmod{p^2}. \end{aligned}$$

In [S6], the author proved the congruences modulo  $p$ .

**Conjecture 3.8** ([S6, Conjecture 5.2]). *Let  $p > 3$  be a prime such that  $p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$ . Then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 U_k(11, 1) \equiv 0 \pmod{p}.$$

#### 4. Conjectures on supercongruences.

In 2003, Rodriguez-Villegas[RV] posed many conjectures on supercongruences. In particular, he conjectured that for any prime  $p > 3$ ,

$$\sum_{k=0}^{p-1} \frac{(6k)!}{1728^k (3k)! k!^3} \equiv \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ with } 2 \nmid x, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{(4k)!}{256^k k!^4} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 3 \mid p-1 \text{ and } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } 3 \mid p-2. \end{cases}$$

The three conjectures have been solved by E. Mortenson[M] and Zhi-Wei Sun[Su2]. Recently my twin brother Zhi-Wei Sun made a lot of conjectures on supercongruences. Inspired by his work in [Su1], the author made the following conjectures.

**Conjecture 4.1** ([S4, Conjecture 2.1]). *Let  $p \equiv 1 \pmod{4}$  be a prime and so  $p = x^2 + y^2$  with  $2 \nmid x$ . Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{648^k k!^4} \equiv 4x^2 - 2p \pmod{p^2}.$$

In [S6], the author proved the congruence modulo  $p$ . When  $p \equiv 3 \pmod{4}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(4k)!}{648^k k!^4} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.2** ([S4, Conjecture 2.2]). *Let  $p \equiv 1 \pmod{3}$  be a prime and so  $p = x^2 + 3y^2$ . Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{(-144)^k k!^4} \equiv 4x^2 - 2p \pmod{p^2}.$$

In [S6], the author proved the congruence modulo  $p$ . When  $p \equiv 5 \pmod{6}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(4k)!}{(-144)^k k!^4} \equiv 0 \pmod{p^2}$ .



**Conjecture 4.3** ([S4, Conjecture 2.3]). *Let  $p \equiv 1, 2, 4 \pmod{7}$  be an odd prime and so  $p = x^2 + 7y^2$ . Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{(-3969)^k k!^4} \equiv 4x^2 - 2p \pmod{p^2}.$$

In [S6], the author proved the congruence modulo  $p$ . When  $p \equiv 3, 5, 6 \pmod{7}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(4k)!}{(-3969)^k k!^4} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.4** ([S4, Conjecture 2.4]). *Let  $p \equiv 1 \pmod{4}$  be a prime and so  $p = x^2 + y^2$  with  $2 \nmid x$ . Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{66^{3k} (3k)! k!^3} \equiv \left(\frac{p}{33}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S8], the author proved the congruence modulo  $p$ . When  $p \equiv 3 \pmod{4}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(6k)!}{66^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.5** ([S4, Conjecture 2.5]). *Let  $p \equiv 1, 3 \pmod{8}$  be a prime and so  $p = x^2 + 2y^2$ . Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{20^{3k} (3k)! k!^3} \equiv \left(\frac{-5}{p}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S8], the author proved the congruence modulo  $p$ . When  $p \equiv 5, 7 \pmod{8}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(6k)!}{20^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.6** ([S4, Conjecture 2.6]). *Let  $p \equiv 1 \pmod{3}$  be a prime and so  $p = x^2 + 3y^2$ . Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{54000^k (3k)! k!^3} \equiv \left(\frac{p}{5}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S8], the author proved the congruence modulo  $p$ . When  $p \equiv 5 \pmod{6}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(6k)!}{54000^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.7** ([S4, Conjecture 2.7]). *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{(6k)!}{(-12288000)^k (3k)! k!^3} \\ & \equiv \begin{cases} \left(\frac{10}{p}\right) (L^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = L^2 + 27M^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

**Conjecture 4.8 ([S4, Conjecture 2.8]).** *Let  $p \equiv 1, 2, 4 \pmod{7}$  be an odd prime and so  $p = x^2 + 7y^2$ . Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{(-15)^{3k} (3k)! k!^3} \equiv \left(\frac{p}{15}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S8], the author proved the congruence modulo  $p$ . When  $p \equiv 3, 5, 6 \pmod{7}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(6k)!}{(-15)^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.9 ([S4, Conjecture 2.9]).** *Let  $p \equiv 1, 2, 4 \pmod{7}$  be an odd prime and so  $p = x^2 + 7y^2$ . Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{255^{3k} (3k)! k!^3} \equiv \left(\frac{p}{255}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S8], the author proved the congruence modulo  $p$ . When  $p \equiv 3, 5, 6 \pmod{7}$ , the author proved  $\sum_{k=0}^{p-1} \frac{(6k)!}{255^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ .

**Conjecture 4.10 ([S4, Conjecture 2.10]).** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{1458^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

**Conjecture 4.11 ([S4, Conjecture 2.11]).** *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{15^{3k}} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and so } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases} \end{aligned}$$

**Conjecture 4.12 ([S4, Conjecture 2.12]).** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-8640)^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 3 \mid p-1, p = x^2 + 3y^2 \text{ and } 10^{\frac{p-1}{3}} \equiv 1 \pmod{p}, \\ p - 2x^2 \pm 6xy \pmod{p^2} & \text{if } 3 \mid p-1, p = x^2 + 3y^2 \text{ and } 10^{\frac{p-1}{3}} \equiv \frac{1}{2}(-1 \mp \frac{x}{y}) \pmod{p}, \\ 0 \pmod{p^2} & \text{if } 3 \mid p-2. \end{cases} \end{aligned}$$

If  $p$  is a prime of the form  $3k + 1$  and so  $p = x^2 + 3y^2$  with  $x, y \in \mathbb{Z}$ , by [S2, Corollary 4.4] we have  $10^{\frac{p-1}{3}} \equiv 1 \pmod{p}$  if and only if  $5 \mid xy$ .

**Conjecture 4.13** ([S6, Conjecture 5.3]). *Let  $p \equiv 1 \pmod{12}$  be a prime and  $p = a^2 + 4b^2 = A^2 + 3B^2$  with  $a \equiv A \equiv 1 \pmod{4}$ . Then*

$$\sum_{k=0}^{(p-1)/6} \frac{\binom{6k}{3k}^2}{(-16)^{3k}} \equiv (-1)^{\frac{p-1}{4}} \frac{1}{3} \left( 2a + 4A - \frac{p}{2a} - \frac{p}{A} \right) \pmod{p^2}.$$

In [S6], the author proved the congruence modulo  $p$ .

For an integer  $m$  and odd prime  $p$  with  $p \nmid m$  let

$$S_p(m) = \sum_{n=0}^{p-1} \sum_{k=0}^n \binom{n}{k}^4 m^k.$$

Recently Zhi-Wei Sun [Su5] investigated congruences  $S_p(m) \pmod{p^2}$  and revealed the connections with binary quadratic forms and series for  $\frac{1}{\pi}$ . Now we introduce the sum

$$Z_p(m) = \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{m^n} \sum_{k=0}^n \binom{n}{k}^3.$$

Then we have the following conjectures concerning  $Z_p(m)$  modulo  $p^2$ .

**Conjecture 4.14.** *Let  $p$  be an odd prime. Then*

$$Z_p(-16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid y, \\ 2p - 4x^2 \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid x - 3, \\ 4\left(\frac{xy}{3}\right)xy \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture 4.15.** *Let  $p$  be an odd prime. Then*

$$Z_p(96) \equiv \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

**Conjecture 4.16.** *Let  $p > 5$  be a prime. Then*

$$Z_p(50) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

**Conjecture 4.17.** *Let  $p > 5$  be a prime. Then*

$$Z_p(16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ and so } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ and so } 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

**Conjecture 4.18.** *Let  $p > 3$  be a prime. Then*

$$Z_p(32) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and so } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

**Conjecture 4.19.** *Let  $p > 7$  be a prime. Then*

$$Z_p(5) \equiv Z_p(-49) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 15y^2 \equiv 1, 19 \pmod{30}, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 5y^2 \equiv 17, 23 \pmod{30}, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 29 \pmod{30}. \end{cases}$$

**Conjecture 4.20.** *Let  $b \in \{7, 11, 19, 31, 59\}$  and*

$$f(b) = \begin{cases} -112 & \text{if } b = 7, \\ -400 & \text{if } b = 11, \\ -2704 & \text{if } b = 19, \\ -24304 & \text{if } b = 31, \\ -1123600 & \text{if } b = 59. \end{cases}$$

*If  $p$  is a prime with  $p \neq 2, 3, b$  and  $p \nmid f(b)$ , then*

$$Z_p(f(b)) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3by^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + by^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } 2p = x^2 + 3by^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } 2p = 3x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-3b}{p}\right) = -1. \end{cases}$$

**Conjecture 4.21.** *Let  $b \in \{5, 7, 13, 17\}$  and*

$$f(b) = \begin{cases} 320 & \text{if } b = 5, \\ 896 & \text{if } b = 7, \\ 10400 & \text{if } b = 13, \\ 39200 & \text{if } b = 17. \end{cases}$$

*If  $p$  is a prime with  $p \neq 2, 3, b$  and  $p \nmid f(b)$ , then*

$$Z_p(f(b)) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6by^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p = 2x^2 + 3by^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 2by^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } p = 6x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6b}{p}\right) = -1. \end{cases}$$

**Conjecture 4.22.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{9n+4}{5^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 4 \left(\frac{p}{5}\right) p \pmod{p^2} \quad \text{for } p > 5, \\ \sum_{n=0}^{p-1} \frac{5n+2}{16^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2p \pmod{p^2}, \\ \sum_{n=0}^{p-1} \frac{9n+2}{50^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2 \left(\frac{-1}{p}\right) p \pmod{p^2} \quad \text{for } p \neq 5, \\ \sum_{n=0}^{p-1} \frac{5n+1}{96^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv \left(\frac{-2}{p}\right) p \pmod{p^2}, \\ \sum_{n=0}^{p-1} \frac{6n+1}{320^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv \left(\frac{p}{15}\right) p \pmod{p^2} \quad \text{for } p \neq 5, \\ \sum_{n=0}^{p-1} \frac{90n+13}{896^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 13 \left(\frac{p}{7}\right) p \pmod{p^2} \quad \text{for } p \neq 7, \\ \sum_{n=0}^{p-1} \frac{102n+11}{10400^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 11 \left(\frac{p}{39}\right) p \pmod{p^2} \quad \text{for } p \neq 5, 13 \end{aligned}$$

**Conjecture 4.23.** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{n=0}^{p-1} \frac{3n+1}{(-16)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv \left(\frac{-1}{p}\right) p \pmod{p^2}, \\ \sum_{n=0}^{p-1} \frac{15n+4}{(-49)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 4 \left(\frac{p}{3}\right) p \pmod{p^2} \quad \text{for } p \neq 5, 7, \\ \sum_{n=0}^{p-1} \frac{9n+2}{(-112)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2 \left(\frac{p}{7}\right) p \pmod{p^2} \quad \text{for } p \neq 7, \\ \sum_{n=0}^{p-1} \frac{99n+17}{(-400)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 17 \left(\frac{-1}{p}\right) p \pmod{p^2}, \\ \sum_{n=0}^{p-1} \frac{855n+109}{(-2704)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 109p \left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } p \neq 13, \\ \sum_{n=0}^{p-1} \frac{585n+58}{(-24304)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 58p \left(\frac{-31}{p}\right) \pmod{p^2} \quad \text{for } p \neq 7, 31. \end{aligned}$$

**Conjecture 4.24.** *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{63k+8}{(-15)^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 8p \left( \frac{-15}{p} \right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{133k+8}{255^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 8p \left( \frac{-255}{p} \right) \pmod{p^2} \quad \text{for } p \neq 17, \\ \sum_{k=0}^{p-1} \frac{28k+3}{20^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 3p \left( \frac{-5}{p} \right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{63k+5}{66^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 5p \left( \frac{-33}{p} \right) \pmod{p^2} \quad \text{for } p \neq 11, \\ \sum_{k=0}^{p-1} \frac{11k+1}{54000^k} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv p \left( \frac{-15}{p} \right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} \frac{506k+31}{(-12288000)^k} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 31p \left( \frac{-30}{p} \right) \pmod{p^2}. \end{aligned}$$

**Conjecture 4.25 ([S7, Conjecture 4.1]).** *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} (-4)^k &\equiv \left( \frac{5}{p} \right) 5^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} (-4)^k \\ &\equiv \begin{cases} \left( \frac{x}{3} \right) (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 15y^2, \\ -\left( \frac{x}{3} \right) (10x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = 5x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{30} \end{cases} \end{aligned}$$

and so

$$2x \left( \frac{x}{3} \right) \equiv \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} (-4)^k \pmod{p} \quad \text{for } p = 5x^2 + 3y^2.$$

**Conjecture 4.26 ([S7, Conjecture 4.2]).** *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{2^k} &\equiv \left( \frac{2}{p} \right) 2^{\frac{(\frac{p}{3})-1}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{2^k} \\ &\equiv \begin{cases} \left( \frac{x}{3} \right) (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ \left( \frac{x}{3} \right) (2x - \frac{p}{4x}) \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 13, 19 \pmod{24} \end{cases} \end{aligned}$$

and so

$$x \left( \frac{x}{3} \right) \equiv -\frac{1}{4} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{2^k} \pmod{p} \quad \text{for } p = 2x^2 + 3y^2.$$

**Conjecture 4.27** ([S7, Conjecture 4.3]). *Let  $p$  be a prime with  $p \neq 2, 3, 17$ . Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{(-16)^k} &\equiv \left(\frac{17}{p}\right) \left(\frac{17}{16}\right)^{\frac{1-\left(\frac{p}{3}\right)}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{(-16)^k} \\ &\equiv \begin{cases} -\left(\frac{x}{3}\right) \left(x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 4p = x^2 + 51y^2, \\ \frac{1}{4} \left(\frac{x}{3}\right) \left(17x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 4p = 17x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{3}\right) = -\left(\frac{p}{17}\right) = 1 \end{cases} \end{aligned}$$

and so

$$x\left(\frac{x}{3}\right) \equiv -\frac{1}{4} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{(-16)^k} \pmod{p} \quad \text{for } 4p = 17x^2 + 3y^2.$$

**Conjecture 4.28** ([S7, Conjecture 4.4]). *Let  $p$  be a prime with  $p \neq 2, 3, 41$ . Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{(-1024)^k} &\equiv \left(\frac{41}{p}\right) \left(\frac{1025}{1024}\right)^{\frac{1-\left(\frac{p}{3}\right)}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{(-1024)^k} \\ &\equiv \begin{cases} -\left(\frac{x}{3}\right) \left(x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 4p = x^2 + 123y^2, \\ \frac{5}{32} \left(\frac{x}{3}\right) \left(41x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 4p = 41x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{3}\right) = -\left(\frac{p}{41}\right) = 1 \end{cases} \end{aligned}$$

and so

$$x\left(\frac{x}{3}\right) \equiv -\frac{5}{32} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{(-1024)^k} \pmod{p} \quad \text{for } 4p = 41x^2 + 3y^2.$$

**Conjecture 4.29** ([S7, Conjecture 4.5]). *Let  $p$  be a prime with  $p \neq 2, 3, 5, 89$ . Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{(-250000)^k} &\equiv \left(\frac{89}{p}\right) \left(\frac{250001}{250000}\right)^{\frac{1-\left(\frac{p}{3}\right)}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{(-250000)^k} \\ &\equiv \begin{cases} -\left(\frac{x}{3}\right) \left(x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 4p = x^2 + 267y^2, \\ \frac{53}{500} \left(\frac{x}{3}\right) \left(89x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 4p = 89x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } \left(\frac{p}{3}\right) = -\left(\frac{p}{89}\right) = 1 \end{cases} \end{aligned}$$

and so

$$x\left(\frac{x}{3}\right) \equiv -\frac{53}{500} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{(-250000)^k} \pmod{p} \quad \text{for } 4p = 89x^2 + 3y^2.$$

**Conjecture 4.30** ([S7, Conjecture 4.6]). *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \frac{1}{(-80)^k} \\ & \equiv \left(\frac{5}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \frac{1}{(-80)^k} \\ & \equiv \begin{cases} x - \frac{p}{x} \pmod{p^2} & \text{if } p \equiv 1, 19 \pmod{30} \text{ and so } 4p = x^2 + 75y^2 \text{ with } 3 \mid x - 2, \\ 5x - \frac{p}{5x} \pmod{p^2} & \text{if } p \equiv 7, 13 \pmod{30} \text{ and so } 4p = 25x^2 + 3y^2 \text{ with } 3 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{30} \end{cases} \end{aligned}$$

**Conjecture 4.31** ([S7, Conjecture 4.7]). *Let  $p > 11$  be a prime such that  $\left(\frac{p}{11}\right) = 1$  and so  $4p = x^2 + 11y^2$ . Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \left(\frac{27}{16}\right)^k & \equiv \left(-\frac{11}{16}\right)^{\frac{1-\left(\frac{p}{3}\right)}{2} p-1} \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \left(\frac{27}{16}\right)^k \\ & \equiv \begin{cases} -\left(\frac{6(-11+x/y)}{p}\right) \left(\frac{x}{11}\right) \left(x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 3 \mid p - 1, \\ -\frac{1}{4} \left(\frac{6(-11+\left(\frac{x}{11}\right)x/y)}{p}\right) (11y - \frac{p}{y}) \pmod{p^2} & \text{if } 3 \mid p - 2 \end{cases} \end{aligned}$$

and so

$$\begin{aligned} & y \left(\frac{-11 + \left(\frac{x}{11}\right)x/y}{p}\right) \\ & \equiv \frac{1}{4} \left(\frac{6}{p}\right)^{(p-5)/6} \sum_{k=0}^{(p-2)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \left(\frac{27}{16}\right)^k \pmod{p} \quad \text{for } 4p = x^2 + 11y^2 \equiv 2 \pmod{3}. \end{aligned}$$

**Conjecture 4.32** ([S7, Conjecture 4.8]). *Let  $p \equiv 1 \pmod{3}$  be a prime and so  $4p = L^2 + 27M^2$  with  $3 \mid L - 2$ . Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \left(-\frac{9}{16}\right)^k \equiv \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \left(-\frac{9}{16}\right)^k \equiv L - \frac{p}{L} \pmod{p^2}.$$

**Conjecture 4.33** ([S7, Conjecture 4.9]). *Let  $p \equiv 1, 3 \pmod{8}$  be a prime and so  $p = c^2 + 2d^2$  with  $4 \mid c - 1$ . Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \left(\frac{27}{2}\right)^k \\ & \equiv \left(-\frac{25}{2}\right)^{\frac{1-\left(\frac{p}{3}\right)}{2} p-1} \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \left(\frac{27}{2}\right)^k \\ & \equiv \begin{cases} (-1)^{\lfloor \frac{p}{8} \rfloor} \left(\frac{-2-c/d}{p}\right) (2c - \frac{p}{2c}) \pmod{p^2} & \text{if } p \equiv 1, 19 \pmod{24}, \\ (-1)^{\lfloor \frac{p}{8} \rfloor} \left(\frac{-2-c/d}{p}\right) (10d - \frac{5p}{4d}) \pmod{p^2} & \text{if } p \equiv 11, 17 \pmod{24}. \end{cases} \end{aligned}$$



**Conjecture 4.34** ([S7, Conjecture 3.1]). *Let  $p$  be a prime such that  $p \equiv 1 \pmod{3}$ ,  $4p = L^2 + 27M^2$  ( $L, M \in \mathbb{Z}$ ) and  $L \equiv 2 \pmod{3}$ . Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 9^k \equiv L - \frac{p}{L} \pmod{p^2}$$

**Conjecture 4.35** ([S7, Conjecture 3.2]). *Let  $p$  be a prime of the form  $4k+1$  and so  $p = x^2 + y^2$  with  $4 \mid x-1$ . Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 (-8)^k \equiv (-1)^{\frac{p-1}{4}} \left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{(-8)^k} \equiv \begin{cases} (-1)^{\frac{y}{4}} \left(2x - \frac{p}{2x}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{y-2}{4}} \left(2y - \frac{p}{2y}\right) \pmod{p^2} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

**Conjecture 4.36** ([S7, Conjecture 3.3]). *Let  $p \equiv 1 \pmod{3}$  be a prime and so  $p = A^2 + 3B^2$ . Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k &\equiv \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} (-8)^k \\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{A-1}{2}} \left(2A - \frac{p}{2A}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ (-1)^{\frac{p+1}{4} + \frac{B-1}{2}} \left(6B - \frac{p}{2B}\right) \pmod{p^2} & \text{if } p \equiv 7 \pmod{12}. \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{4^k} \equiv \begin{cases} (-1)^{\frac{A-1}{2}} \left(2A - \frac{p}{2A}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ (-1)^{\frac{B-1}{2}} \left(3B - \frac{p}{4B}\right) \pmod{p^2} & \text{if } p \equiv 7 \pmod{12}. \end{cases}$$

**Conjecture 4.37** ([S7, Conjecture 3.4]). *Let  $p > 2$  be a prime such that  $p \equiv 1, 2, 4 \pmod{7}$  and so  $p = x^2 + 7y^2$ . Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 64^k \equiv \begin{cases} \left(\frac{2}{p}\right) (-1)^{\frac{x-1}{2}} \left(2x - \frac{p}{2x}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \left(\frac{2}{p}\right) (-1)^{\frac{y-1}{2}} \left(42y - \frac{3p}{2y}\right) \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{64^k} \equiv \begin{cases} (-1)^{\frac{x-1}{2}} \left(2x - \frac{p}{2x}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{3}{4} (-1)^{\frac{y-1}{2}} \left(7y - \frac{p}{4y}\right) \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture 4.38** ([S7, Conjecture 3.5]). *Let  $p$  be an odd prime. Then*

$$\sum_{k=0}^{p-1} (-1)^k \binom{-\frac{1}{4}}{k}^2 \equiv \begin{cases} (-1)^{\frac{x+y+1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1 \pmod{8}, \\ (-1)^{\frac{y-1}{2}} (4y - \frac{p}{2y}) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 3 \pmod{8}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

For the results related to Conjectures 4.25-4.38, see [S7].

**Conjecture 4.39** ([S7, Conjecture 3.6]). *Let  $p$  be an odd prime.*

(i) *If  $p \equiv 1 \pmod{4}$  and so  $p = x^2 + y^2$  with  $2 \nmid x$ , then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{4^k} \equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{6}}{k} 2^k$$

$$\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{x+1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and } 3 \mid x, \\ (-1)^{\frac{p-1}{4} + \frac{x-1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and } 3 \mid y, \\ (-1)^{\frac{x-1+y}{2}} (2y - \frac{p}{2y}) \pmod{p^2} & \text{if } 12 \mid p-5 \text{ and } x \equiv y \equiv 1 \pmod{3}. \end{cases}$$

(ii) *If  $p \equiv 3 \pmod{4}$ , then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{4^k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{6}}{k} 2^k \equiv 0 \pmod{p}.$$

**Conjecture 4.40** ([S7, Conjecture 3.7]). *Let  $p > 3$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{(-3)^k} \equiv \left(\frac{3}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{81^k}$$

$$\equiv \begin{cases} \left(\frac{6}{p}\right) (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 4 \mid x-1, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture 4.41** ([S7, Conjecture 3.8]). *Let  $p \neq 2, 5$  be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{(-80)^k}$$

$$\equiv \begin{cases} (-1)^{\binom{\frac{x}{5}}{x-1+y}/2} \left(\frac{x}{5}\right) (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ and } 10 \mid y, \\ (-1)^{(x+1+y)/2} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ and } 10 \mid x-5, \\ (-1)^{(x+1+y)/2} (2y - \frac{p}{2y}) \pmod{p^2} & \text{if } p = x^2 + y^2, 2 \nmid x \\ & \text{and } x \equiv y \equiv 1, 2 \pmod{5}, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

**Conjecture 4.42** ([S7, Conjecture 3.9]). *Let  $p > 2$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} 2^k \\ &= \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 2y^2 \text{ with } x \equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

**Conjecture 4.43** ([S7, Conjecture 3.10]). *Let  $p > 7$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k} (-3)^k \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{(-27)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{2}{3}}{k}}{(-4)^k} \\ & \equiv \binom{p}{5} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{5^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k} 2^k \\ & \equiv \begin{cases} 2A - \frac{p}{2A} \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } 3 \mid A - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

**Conjecture 4.44** ([S7, Conjecture 3.11]). *Let  $p > 3$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{(-4)^k} \\ &= \begin{cases} \binom{p}{5} (2A - \frac{p}{2A}) \pmod{p^2} \\ \quad \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } 5 \mid AB \text{ and } 3 \mid A - 1, \\ -\binom{p}{5} (A + 3B - \frac{p}{A+3B}) \pmod{p^2} \\ \quad \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } A/B \equiv -1, -2 \pmod{5} \text{ and } 3 \mid A - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

**Conjecture 4.45** ([S7, Conjecture 3.12]). *Let  $p > 5$  be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{6}}{k} \left(-\frac{3}{125}\right)^k \\ & \equiv \begin{cases} \binom{-5}{p} \binom{A}{3} (2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

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